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An sl(4, ℝ) Lie algebraic treatment of the first family of Pöschl–Teller potentials

C Quesne†

Service de Physique Théorique et Mathématique, CP 229, Université Libre de Bruxelles, Bd du Triomphe, B1050 Brussels, Belgium

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Abstract. An sl(4, ℝ) dynamical potential algebra, containing the so(4) potential algebra, is constructed for the two-parameter Pöschl–Teller potentials of the first kind. For this purpose, the relation between the Wigner rotation matrices and the solutions of the first Pöschl–Teller equation is used. Explicit expressions are given for the sl(4, ℝ) generators, as well as for their action on the normalised solutions. All the Hamiltonian eigenstates, corresponding to the family of potentials with quantised potential strengths (m', m) differing by integers, are proved to belong to a single sl(4, ℝ) unitary irreducible representation of the ladder series. For integral values of m' and m, the latter is $\mathfrak{D}^{\text{ladd}}(0, 0; \eta)$, while, for half-integral values, it is $\mathfrak{D}^{\text{ladd}}(\frac{1}{2}, \frac{1}{2}; \eta)$, where η is some real parameter. Both irreducible representations are also shown to be characterised by generalised Young pattern labels $[pqr0]$, where $p = -2 - \frac{1}{2}\mathrm{i}\eta$, and $q = r = 0$.

1. Introduction

Group theoretical methods based upon dynamical (invariance or non-invariance) algebras have been successfully used in various quantum mechanical problems, such as the harmonic oscillator and Coulomb ones (Wybourne 1974 and references therein). Following the ideas introduced by Gell-Mann in particle physics (Dothan *et al* 1965), by a dynamical non-invariance algebra one means a Lie algebra such that the Hamiltonian of the system under consideration can be expressed as a function of its generators. All the Hamiltonian eigenstates then fall into one (or at most a few) irreducible representation (irrep) of the algebra, while the generators of the latter connect together the eigenstates.

Recently, a second type of non-invariance algebras, called potential algebras, was introduced in the study of the Morse potential and of a one-parameter Pöschl–Teller potential of the second kind (Alhassid *et al* 1983, 1986, Frank and Wolf 1984). These algebras are such that their irrep carrier spaces contain states with the same energy, but corresponding to different quantised potential strengths. The Hamiltonian of the potential family is then essentially the Casimir operator of the algebra. Later on, potential algebras were also found for the two-parameter Pöschl–Teller potentials of the first and second kinds, and for the Rosen–Morse and Eckart potentials (Frank and Wolf 1985, Barut *et al* 1987a, b).

† Directeur de recherches FNRS.

In the cases of the Morse potential and of a one-parameter Pöschl–Teller potential of the second kind, a third type of non-invariance algebras, combining features of both the dynamical and potential ones, was considered (Alhassid *et al* 1983). Such more general algebras, whose generators can connect together both states with the same potential strength but different energies, and states with the same energy but different quantised potential strengths (as well as states with different potential strengths and energies), may be called dynamical potential algebras.

An interesting question is whether a dynamical potential algebra can be constructed for the other exactly solvable one-dimensional problems for which a potential algebra was shown to exist. In the conclusion of their papers on the two-parameter Pöschl–Teller, Rosen–Morse, and Eckart potentials, Barut *et al* (1987a, b) answer this question positively and assert that so(4, 2) is the searched-for algebra.

The purpose of the present paper is to critically examine this statement. For the sake of demonstration, restricting ourselves to the first family of Pöschl–Teller potentials, we shall prove that the so(4, 2) algebra, containing its so(4) potential algebra, does not fulfil the conditions required for a dynamical potential algebra, and we shall propose sl(4, $\mathbb{R}$) as an alternative choice.

In § 2, we review the relation between the solutions of the first Pöschl–Teller equation and the Wigner rotation matrices. In § 3, we use this connection to obtain the so(4) potential algebra of the Pöschl–Teller potentials, and show that so(4, 2) is not suitable as a dynamical potential algebra. In § 4, we build the sl(4, $\mathbb{R}$) dynamical algebra of the rotation matrices, study its irreps, and then combine these results with those of § 2 to prove that sl(4, $\mathbb{R}$) is a dynamical potential algebra for the first family of Pöschl–Teller potentials, and to give explicit expressions for its generators and for their action on the wavefunctions. Finally, § 5 contains the conclusion.

2. The solutions of the first Pöschl–Teller equation in terms of rotation matrices

The first Pöschl–Teller equation (Pöschl and Teller 1933) is

$$\left[-\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\hbar^2a^2}{2M}\left(\frac{\kappa(\kappa-1)}{\sin^2(ax)}+\frac{\lambda(\lambda-1)}{\cos^2(ax)}\right)-E_n\right]\psi_n(x)=0 \qquad \kappa,\lambda>1 \tag{2.1}$$

where a defines the range of the variable x ($x\in[0,\pi/2a]$), κ and λ are two strength parameters, and $n\in\mathbb{N}$ labels the eigenvalues E_n and the wavefunctions $\psi_n(x)$. It is convenient to replace κ and λ by m' and m, defined by

$$\kappa=m'+m+\tfrac{1}{2} \qquad \lambda=m'-m+\tfrac{1}{2} \tag{2.2}$$

and to write the wavefunctions as $\psi_n^{(m',m)}(x)$. The condition $\kappa,\lambda>1$ imposes the following restriction on m' and m:

$$m'>|m|+\tfrac{1}{2}. \tag{2.3}$$

Following the analysis of Barut *et al* (1987a), let us set

$$x=(\pi-\beta)/2a \qquad \beta\in[0,\pi] \tag{2.4}$$

$$E_n=2\hbar^2a^2\Lambda_n/M \tag{2.5}$$

and

$$\psi_n^{(m',m)}(x)=[(2j+1)a\sin\beta]^{1/2}\varphi_n^{(m',m)}(\beta) \tag{2.6}$$

where j will be defined below in terms of m' and n. Equation (2.1) is then transformed into the following equation:

$$[-d^2_{\beta\beta} - \cot\beta\, d_\beta + (m'^2 + m^2 - 2m'm\cos\beta)\operatorname{cosec}^2\beta - \Lambda_n + \tfrac{1}{4}]\varphi_n^{(m',m)}(\beta) = 0. \tag{2.7}$$

The latter coincides with the differential equation satisfied by the β-dependent part $d^j_{m'm}(\beta)$ of the Wigner rotation matrices (Biedenharn and Louck 1981), provided that $\Lambda_n = (j+\frac{1}{2})^2$, where j is integral or half-integral, and $j-|m'|, j-|m| \in \mathbb{N}$. From (2.2) and (2.3), these conditions imply that κ and λ must be half-integral, and that

$$j = m' + n \qquad n \in \mathbb{N}. \tag{2.8}$$

Hence, the eigenvalues can be written in dimensionless units as

$$\Lambda_n = (m' + n + \tfrac{1}{2})^2 = \tfrac{1}{4}(\kappa + \lambda + 2n)^2 \qquad n \in \mathbb{N}. \tag{2.9}$$

The corresponding normalised wavefunctions are given by equation (2.6), where

$$\varphi_n^{(m',m)}(\beta) = d^j_{m'm}(\beta). \tag{2.10}$$

By introducing an additional dependence on two auxiliary angular variables $\alpha, \gamma \in [0, 2\pi)$ (Barut *et al* 1987a), the wavefunctions (2.6) are transformed into the extended wavefunctions

$$\begin{aligned}\Psi_n^{(m',m)}(x,\alpha,\gamma) &= (2\pi)^{-1}\exp(\mathrm{i}m'\alpha)\psi_n^{(m',m)}(x)\exp(\mathrm{i}m\gamma)\\ &= [(2j+1)a/4\pi^2]^{1/2}(\sin\beta)^{1/2} D^{j*}_{m'm}(\alpha,\beta,\gamma)\end{aligned} \tag{2.11}$$

where $D^j_{m'm}(\alpha,\beta,\gamma)$ is a rotation matrix element written in terms of Euler angles (Biedenharn and Louck 1981). Since, for fixed j, the functions $D^{j*}_{m'm}(\alpha,\beta,\gamma)$ form a basis for an so(4) irrep, the same is true for the functions $\Psi_n^{(m',m)}(x,\alpha,\gamma)$. As first shown by Barut *et al* (1987a), so(4) is therefore a potential algebra for the first family of Pöschl–Teller potentials.

Before proceeding to construct this potential algebra in the next section, an important property, left unstressed by Barut *et al* (1987a), is worth emphasising. From (2.3), it indeed follows that there is no one-to-one correspondence between the functions $d^j_{m'm}(\beta)$, $-j \le m', m \le j$, and the Pöschl–Teller potential wavefunctions $\psi_n^{(m',m)}(x)$, nor between the complex conjugate rotation matrix elements $D^{j*}_{m'm}(\alpha,\beta,\gamma)$, $-j \le m', m \le j$, and the extended wavefunctions $\Psi_n^{(m',m)}(x,\alpha,\gamma)$.

To restore bijectiveness, we may try to enlarge the family of Pöschl–Teller potentials. From (2.1), we note that the four sets of parameters (κ,λ), $(\kappa, 1-\lambda)$, $(1-\kappa,\lambda)$ and $(1-\kappa, 1-\lambda)$—or (m',m), (m,m'), $(-m,-m')$ and $(-m',-m)$—correspond to the same potential. Hence, we may extend the range of definition of κ and λ by considering the values $(\kappa > 1, \lambda < 0)$, $(\kappa < 0, \lambda > 1)$ and $(\kappa, \lambda < 0)$, in addition to $(\kappa, \lambda > 1)$. Such values of κ and λ respectively correspond to values of m' and m satisfying the conditions $m > |m'| + \frac{1}{2}$, $m < -|m'| - \frac{1}{2}$, $m' < -|m| - \frac{1}{2}$, and (2.3). However, owing to well known symmetry properties of $d^j_{m'm}(\beta)$ (Biedenharn and Louck 1981), the three sets of functions resulting from this extension are but replicas of the set of wavefunctions (2.6), namely

$$\psi_n^{(m',m)}(x) = (-1)^{m'-m}\psi_n^{(m,m')}(x) = \psi_n^{(-m,-m')}(x) = (-1)^{m'-m}\psi_n^{(-m',-m)}(x). \tag{2.12}$$

The values $\kappa = \frac{1}{2}$ and $\lambda = \frac{1}{2}$, corresponding to $m' = -m$ and $m' = m$ respectively, do not fulfil the above-mentioned conditions. For such parameter values, it can be easily seen that a solution of the Pöschl–Teller equation, by the non-algebraic method previously used in the case $\kappa, \lambda > 1$ (Flügge 1971), cannot be obtained. This is not surprising since, for $\kappa = \frac{1}{2}$ ($\lambda = \frac{1}{2}$), the Pöschl–Teller potential behaves as $-x^{-2}$ $[-(x-\pi/2a)^{-2}]$ for $x \sim 0$ $(x \sim \pi/2a)$, and some problems are known to arise for such highly singular potentials (Case 1950). Hence, the family of Pöschl–Teller potentials cannot be enlarged so that $\psi_n^{(-m,m)}(x)$ and $\psi_n^{(m,m)}(x)$ are wavefunctions corresponding to some potential of the family. Such functions must therefore be considered as unphysical.

In conclusion, a relation has been obtained between the solutions of the first Pöschl–Teller equation and the rotation matrices at the cost of adding to the former three replicas of the whole set, as well as some unphysical functions.

3. The so(4) potential algebra of the Pöschl–Teller potentials and its embedding into so(4, 2)

The Barut *et al* (1987a) procedure for deriving the Pöschl–Teller potential algebra was based upon the algebraic version (Miller 1964, 1968, Kaufman 1966) of the factorisation method (Infeld and Hull 1951). However, the latter proves to be unsuitable as a starting point for the construction of the semisimple dynamical potential algebra that we shall carry out in the next section. The ladder operators, raising or lowering the energy eigenvalue for a given potential, indeed result from a type E factorisation, which, according to Miller's approach, can be only indirectly algebraised giving rise to a non-semisimple Euclidean algebra. We will therefore use an alternative procedure, relying on some known properties of the rotation matrices.

Let us start with a brief review of the so(4) Lie algebraic approach to the rotation matrices. As is well known, the complex conjugate rotation matrices $D^{j*}_{m'm}$ can be obtained from the solid hyperspherical harmonics (Biedenharn and Louck 1981), i.e. the homogeneous solutions of degree $N = 2j$ of the four-dimensional Laplace equation:

$$\nabla^2 \mathcal{Y}_{Nm'm}(\boldsymbol{u}) = 0 \tag{3.1}$$

$$\boldsymbol{u} \cdot \nabla \mathcal{Y}_{Nm'm}(\boldsymbol{u}) = N \mathcal{Y}_{Nm'm}(\boldsymbol{u}). \tag{3.2}$$

Here $\boldsymbol{u}$ denotes the set of coordinates u_μ, $\mu = 1, \ldots, 4$, $\nabla^2 = \partial_\mu \partial_\mu$, and $\boldsymbol{u} \cdot \nabla = u_\mu \partial_\mu$, where $\partial_\mu = \partial/\partial u_\mu$, and there is a summation over dummy indices.

If we factor out the homogeneous part as follows:

$$\mathcal{Y}_{Nm'm}(\boldsymbol{u}) = u^N Y_{Nm'm}(\boldsymbol{u}) \qquad u \equiv (u_\mu u_\mu)^{1/2} \tag{3.3}$$

then the remaining factor, the hyperspherical harmonic $Y_{Nm'm}(\boldsymbol{u})$, is defined on the unit sphere S^3. By applying the transformation

$$\begin{aligned} u_1 &= u \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}(\gamma - \alpha) \qquad & u_2 &= u \sin \tfrac{1}{2}\beta \cos \tfrac{1}{2}(\gamma - \alpha) \\ u_3 &= u \cos \tfrac{1}{2}\beta \sin \tfrac{1}{2}(\gamma + \alpha) \qquad & u_4 &= u \cos \tfrac{1}{2}\beta \cos \tfrac{1}{2}(\gamma + \alpha) \end{aligned} \tag{3.4}$$

$Y_{Nm'm}(\boldsymbol{u})$ can be rewritten in terms of Euler angles as follows:

$$Y_{Nm'm}(\boldsymbol{u}) = Y_{Nm'm}(\alpha, \beta, \gamma) = (-1)^{j-m}[(2j+1)/2\pi^2]^{1/2} D^{j*}_{m',-m}(\alpha, \beta, \gamma). \tag{3.5}$$

From (3.1)–(3.3), the hyperspherical harmonics $Y_{Nm'm}(\boldsymbol{u})$ satisfy the equation

$$\Phi Y_{Nm'm}(\boldsymbol{u}) = N(N+2) Y_{Nm'm}(\boldsymbol{u}) \tag{3.6}$$

where

$$\Phi = \tfrac{1}{2} L_{\mu\nu} L_{\mu\nu} = -\boldsymbol{u}^2 \nabla^2 + (\boldsymbol{u} \cdot \nabla)^2 + 2\boldsymbol{u} \cdot \nabla \tag{3.7}$$

is the Casimir operator of the so(4) algebra, whose generators $L_{\mu\nu} = -L_{\nu\mu} = (L_{\mu\nu})^\dagger$ are defined by

$$L_{\mu\nu} = -\mathrm{i}(u_\mu \partial_\nu - u_\nu \partial_\mu) \tag{3.8}$$

and satisfy the commutation relations

$$[L_{\mu\nu}, L_{\mu'\nu'}] = \mathrm{i}(\delta_{\mu\mu'} L_{\nu\nu'} - \delta_{\mu\nu'} L_{\nu\mu'} - \delta_{\nu\mu'} L_{\mu\nu'} + \delta_{\nu\nu'} L_{\mu\mu'}). \tag{3.9}$$

The $(N+1)^2$ functions $Y_{Nm'm}(\boldsymbol{u})$, corresponding to a given N value, form a basis for an so(4) irrep $[N0]$. The indices m' and m $(m', m = -\frac{1}{2}N, -\frac{1}{2}N+1, \ldots, \frac{1}{2}N)$ label the row of this irrep, or, more precisely, the row of the irrep (j, j) of the isomorphic su(2)⊕su(2) algebra, corresponding to the chain su(2)⊕su(2) ⊃ u(1)⊕u(1). The su(2)⊕su(2) generators $J_i = J_i^\dagger$, $K_i = K_i^\dagger$ are defined in terms of $L_{\mu\nu}$ by

$$J_i = \tfrac{1}{2}(\tfrac{1}{2}\varepsilon_{ijk} L_{jk} - L_{i4}) \qquad K_i = \tfrac{1}{2}(\tfrac{1}{2}\varepsilon_{ijk} L_{jk} + L_{i4}) \tag{3.10}$$

where Latin indices run over 1, 2, 3, and ε_{ijk} is the antisymmetric tensor. They satisfy the commutation relations

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk} J_k \qquad [K_i, K_j] = \mathrm{i}\varepsilon_{ijk} K_k \qquad [J_i, K_j] = 0 \tag{3.11}$$

and their action on the functions $Y_{Nm'm}(\boldsymbol{u})$ is that of standard angular momentum operators.

By using (3.4), (3.7), (3.8) and (3.10), the operators

$$J_0 = J_3 \qquad J_\pm = J_1 \pm \mathrm{i} J_2 \qquad K_0 = K_3 \qquad K_\pm = K_1 \pm \mathrm{i} K_2 \tag{3.12}$$

and the so(4) Casimir operator Φ can be rewritten in terms of the Euler angles as follows:

$$\begin{aligned} J_0 &= -\mathrm{i}\partial_\alpha & J_\pm &= \mathrm{e}^{\pm\mathrm{i}\alpha}(\mathrm{i}\cot\beta\,\partial_\alpha \pm \partial_\beta - \mathrm{i}\,\mathrm{cosec}\,\beta\,\partial_\gamma) \\ K_0 &= \mathrm{i}\partial_\gamma & K_\pm &= \mathrm{e}^{\mp\mathrm{i}\gamma}(-\mathrm{i}\,\mathrm{cosec}\,\beta\,\partial_\alpha \mp \partial_\beta + \mathrm{i}\cot\beta\,\partial_\gamma) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \Phi &= 4\boldsymbol{J}^2 = 4\boldsymbol{K}^2 \\ &= 4[-\partial^2_{\beta\beta} - \cot\beta\,\partial_\beta - \mathrm{cosec}^2\,\beta(\partial^2_{\alpha\alpha} - 2\cos\beta\,\partial^2_{\alpha\gamma} + \partial^2_{\gamma\gamma})]. \end{aligned} \tag{3.14}$$

From (3.5), their action on the complex conjugate rotation matrices is given by

$$\begin{aligned} J_0 D^{j*}_{m'm}(\alpha, \beta, \gamma) &= m' D^{j*}_{m'm}(\alpha, \beta, \gamma) \\ J_\pm D^{j*}_{m'm}(\alpha, \beta, \gamma) &= [(j \mp m')(j \pm m' + 1)]^{1/2} D^{j*}_{m'\pm1,m}(\alpha, \beta, \gamma) \\ K_0 D^{j*}_{m'm}(\alpha, \beta, \gamma) &= -m D^{j*}_{m'm}(\alpha, \beta, \gamma) \\ K_\pm D^{j*}_{m'm}(\alpha, \beta, \gamma) &= -[(j \pm m)(j \mp m + 1)]^{1/2} D^{j*}_{m',m\mp1}(\alpha, \beta, \gamma) \end{aligned} \tag{3.15}$$

and

$$\boldsymbol{J}^2 D^{j*}_{m'm}(\alpha, \beta, \gamma) = \boldsymbol{K}^2 D^{j*}_{m'm}(\alpha, \beta, \gamma) = j(j+1) D^{j*}_{m'm}(\alpha, \beta, \gamma). \tag{3.16}$$

After these preliminaries, it is now straightforward to construct the so(4) Pöschl-Teller potential algebra. From (2.11), it follows that its generators $\tilde{L}_{\mu\nu}$ can be obtained from $L_{\mu\nu}$ by the similarity transformation

$$\tilde{L}_{\mu\nu} = (\sin\beta)^{1/2} L_{\mu\nu} (\sin\beta)^{-1/2} \tag{3.17}$$

combined with the change of variables (2.4), and that the same is true for the generators $\tilde{J}_i$ and $\tilde{K}_i$ of the isomorphic su(2)⊕su(2) algebra. The results can be written as

$$\begin{aligned}
\tilde{J}_0 &= -\mathrm{i}\partial_\alpha \\
\tilde{J}_\pm &= \mathrm{e}^{\pm\mathrm{i}\alpha}[\mp(2a)^{-1}\partial_x - \mathrm{i}\cot(2ax)\partial_\alpha - \mathrm{i}\,\mathrm{cosec}(2ax)\partial_\gamma \pm \tfrac{1}{2}\cot(2ax)] \\
\tilde{K}_0 &= \mathrm{i}\partial_\gamma \\
\tilde{K}_\pm &= \mathrm{e}^{\mp\mathrm{i}\gamma}[\pm(2a)^{-1}\partial_x - \mathrm{i}\,\mathrm{cosec}(2ax)\partial_\alpha - \mathrm{i}\cot(2ax)\partial_\gamma \mp \tfrac{1}{2}\cot(2ax)].
\end{aligned} \tag{3.18}$$

As can be easily checked, the operators (3.13) satisfy the hermiticity properties $J_0^\dagger = J_0$, $J_\pm^\dagger = J_\mp$, $K_0^\dagger = K_0$, $K_\pm^\dagger = K_\mp$ with respect to the measure $\sin\beta\, \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}\gamma$, while the operators (3.18) have similar properties when the measure is replaced by $\mathrm{d}x\, \mathrm{d}\alpha\, \mathrm{d}\gamma$, as it should be.

From (2.8), the action of the operators (3.18) on the extended wavefunctions (2.11) is given by

$$\begin{aligned}
\tilde{J}_0\Psi_n^{(m',m)}(x,\alpha,\gamma) &= m'\Psi_n^{(m',m)}(x,\alpha,\gamma) \\
\tilde{J}_+\Psi_n^{(m',m)}(x,\alpha,\gamma) &= [n(2m'+n+1)]^{1/2}\Psi_{n-1}^{(m'+1,m)}(x,\alpha,\gamma) \\
\tilde{K}_0\Psi_n^{(m',m)}(x,\alpha,\gamma) &= -m\Psi_n^{(m',m)}(x,\alpha,\gamma) \\
\tilde{K}_+\Psi_n^{(m',m)}(x,\alpha,\gamma) &= -[(m'+m+n)(m'-m+n+1)]^{1/2}\Psi_n^{(m',m-1)}(x,\alpha,\gamma)
\end{aligned} \tag{3.19}$$

and similar relations for $\tilde{J}_-$ and $\tilde{K}_-$, resulting from their hermiticity properties. Hence, the generators of the potential algebra so(4) ≃ su(2)⊕su(2) connect together the eigenstates associated with the same eigenvalue Λ_n, given by (2.9), but with different potentials corresponding to the sets of quantised potential strengths (m', m), $(m'\pm1, m)$ and $(m', m\pm1)$. All such states belong to a single so(4) irrep $[N0]$, where $N = 2m' + 2n$. Moreover, after substituting $-\mathrm{i}\partial_\alpha$ and $-\mathrm{i}\partial_\gamma$ for m' and m respectively, the Pöschl–Teller Hamiltonian H, as defined in (2.1), is essentially the so(4) Casimir operator, since

$$\tilde{\boldsymbol{J}}^2 = \tilde{\boldsymbol{K}}^2 = M(2\hbar^2a^2)^{-1}H - \tfrac{1}{4}. \tag{3.20}$$

We would like now to enlarge the so(4) algebra by including some operators changing Λ_n, but keeping m' and m fixed. If they are to generate the whole spectrum corresponding to a given potential, from (2.8) they must raise or lower j by one unit, or, in other words, give rise to transitions between the so(4) irreps $[N0]$ and $[N\pm2, 0]$, or between the su(2)⊕su(2) irreps (j, j) and $(j\pm1, j\pm1)$. Hence, they must transform under the so(4) irrep [20], or the su(2)⊕su(2) irrep (1, 1).

As suggested by Barut *et al* (1987a), so(4) can be extended to the non-compact algebra so(4, 2). However, the extra generators separate into an so(4) scalar, namely the so(2) generator, and two irreducible tensors transforming under the so(4) ≃ su(2)⊕su(2) irrep $[10] \simeq (\tfrac{1}{2}, \tfrac{1}{2})$. Hence, so(4, 2) is not a suitable candidate for the dynamical potential algebra of the Pöschl–Teller potentials.

The so(4) algebra can also be embedded into sl(4, ℝ) ≃ so(3, 3) (Dothan and Ne'eman 1966, Ne'eman and Šijački 1979, Šijački and Ne'eman 1985). This time, the extra generators transform under the so(4) irrep [20], so that sl(4, ℝ) provides us with a dynamical potential algebra for the Pöschl–Teller potentials. In the next section, we shall first review the construction of the sl(4, ℝ) dynamical algebra of the rotation matrices, then apply our results to the Pöschl–Teller potentials.

4. The sl(4, ℝ) dynamical potential algebra of the Pöschl–Teller potentials

From the general study of the multiplicity-free unitary irreps of $\overline{\mathrm{SL}}(4,\mathbb{R})$, the double-covering group of SL(4, ℝ) (Ne'eman and Šijački 1979, Friedman and Sorkin 1980, Šijački and Ne'eman 1985), it is known that sl(4, ℝ) may be considered as a dynamical algebra for the rotation matrices. The relevant irreps actually belong to the ladder series. Since, unfortunately, there is some confusion about the $\overline{\mathrm{SL}}(4,\mathbb{R})$ irreps in the literature, we feel it useful to carry out in the present section an independent and self-contained construction of the sl(4, ℝ) irreps, relevant to the study of the rotation matrices, in a form directly applicable to the Pöschl–Teller potentials.

The sl(4, ℝ) algebra is generated by the so(4) operators $L_{\mu\nu}$, introduced in § 3, and by some extra operators $S_{\mu\nu}=S_{\nu\mu}=(S_{\mu\nu})^{\dagger}$, $\mu,\nu=1,\ldots,4$, such that $S_{\mu\mu}=0$. Their commutation relations are given by (3.9), and by

$$\begin{aligned}[L_{\mu\nu},S_{\mu'\nu'}]&=\mathrm{i}(\delta_{\mu\mu'}S_{\nu\nu'}+\delta_{\mu\nu'}S_{\nu\mu'}-\delta_{\nu\mu'}S_{\mu\nu'}-\delta_{\nu\nu'}S_{\mu\mu'})\\ [S_{\mu\nu},S_{\mu'\nu'}]&=-\mathrm{i}(\delta_{\mu\mu'}L_{\nu\nu'}+\delta_{\mu\nu'}L_{\nu\mu'}+\delta_{\nu\mu'}L_{\mu\nu'}+\delta_{\nu\nu'}L_{\mu\mu'}).\end{aligned}\tag{4.1}$$

In the Cartan decomposition of sl(4, ℝ), so(4) is the maximal compact subalgebra, while the nine independent non-compact generators $S_{\mu\nu}$ belong to the orthogonal complementary subspace.

Note that sl(4, ℝ) is isomorphic to the so(3, 3) algebra, whose generators $\Lambda_{AB}=-\Lambda_{BA}=(\Lambda_{AB})^{\dagger}$, $A,B=1,\ldots,6$, may be defined by

$$\begin{aligned}\Lambda_{ij}&=\varepsilon_{ijk}J_k\\ \Lambda_{3+i,3+j}&=-\varepsilon_{ijk}K_k\\ \Lambda_{i,3+j}&=-\Lambda_{3+j,i}=\tfrac{1}{2}(S_{ij}+\delta_{ij}S_{44}-\varepsilon_{ijk}S_{k4})\end{aligned}\tag{4.2}$$

where, as before, Latin indices run over 1, 2, 3. Their commutation relations are given by

$$[\Lambda_{AB},\Lambda_{CD}]=\mathrm{i}(g_{AC}\Lambda_{BD}-g_{AD}\Lambda_{BC}-g_{BC}\Lambda_{AD}+g_{BD}\Lambda_{AC})\tag{4.3}$$

where the metric tensor is $g_{AB}=\mathrm{diag}(+1,+1,+1,-1,-1,-1)$. The operators Λ_{ij} and $\Lambda_{3+i,3+j}$ generate the maximal compact subalgebra so(3)⊕so(3). The corresponding groups SL(4, ℝ) and SO(3, 3) satisfy the isomorphism relation $\mathrm{SL}(4,\mathbb{R})/\mathrm{Z}_2\simeq\mathrm{SO}(3,3)$, where Z_2 is a two-element subgroup of SL(4, ℝ).

Instead of the sl(4, ℝ) generators $L_{\mu\nu}$ and $S_{\mu\nu}$ (with $S_{\mu\mu}=0$), it is convenient to use the su(2)⊕su(2) generators J_i, K_i, or J_0, $J_{\pm}$, K_0, $K_{\pm}$, defined in (3.10) and (3.12) respectively, and the components $T_{\sigma\tau}$, $\sigma,\tau=+1,0,-1$ of an irreducible tensor of rank (1, 1) with respect to su(2)⊕su(2). Its highest weight component is defined by

$$T_{+1,+1}=\tfrac{1}{2}(\Lambda_{14}+\mathrm{i}\Lambda_{15}+\mathrm{i}\Lambda_{24}-\Lambda_{25})\tag{4.4}$$

and the remaining ones can be obtained from the commutation relations

$$\begin{aligned}&[J_0,T_{\sigma\tau}]=\sigma T_{\sigma\tau} &&[J_{\pm},T_{\sigma\tau}]=[(1\mp\sigma)(2\pm\sigma)]^{1/2}T_{\sigma\pm1,\tau}\\ &[K_0,T_{\sigma\tau}]=\tau T_{\sigma\tau} &&[K_{\pm},T_{\sigma\tau}]=[(1\mp\tau)(2\pm\tau)]^{1/2}T_{\sigma,\tau\pm1}.\end{aligned}\tag{4.5}$$

Those of two operators $T_{\sigma\tau}$ and $T_{\sigma'\tau'}$ are given by

$$\begin{aligned}[T_{\sigma\tau},T_{\sigma'\tau'}]=&(-1)^{\tau}\delta_{\tau,-\tau'}\sqrt{2}\langle 1\ \sigma,1\ \sigma'|1\ \sigma+\sigma'\rangle J_{\sigma+\sigma'}\\ &+(-1)^{\sigma}\delta_{\sigma,-\sigma'}\sqrt{2}\langle 1\ \tau,1\ \tau'|1\ \tau+\tau'\rangle K_{\tau+\tau'}\end{aligned}\tag{4.6}$$

where $\langle\ ,\ |\ \rangle$ denotes an SU(2) Wigner coefficient, $J_{\pm1}=\mp J_{\pm}/\sqrt{2}$, and $K_{\pm1}=\mp K_{\pm}/\sqrt{2}$.

On the unit sphere S^3, where the so(4) generators $L_{\mu\nu}$ are realised by the first-order differential operators (3.8), the non-compact generators $S_{\mu\nu}$ can be realised by the operators

$$S_{\mu\nu}=(1/2\mathrm{i})[\Phi, u_\mu u_\nu]+\eta(u_\mu u_\nu-\tfrac{1}{4}\delta_{\mu\nu}) \tag{4.7}$$

obtained by applying Gell-Mann's decontraction procedure (Dothan *et al* 1965, Dothan and Ne'eman 1966). In (4.7), Φ is the so(4) Casimir operator, defined in (3.7), and η is a parameter, which may take any real value (including zero) and will serve to label the sl(4, $\mathbb{R}$) irreps. Note that the operators (4.7) are also first-order differential operators, since they can be rewritten as

$$S_{\mu\nu}=u_\rho u_\mu L_{\nu\rho}+u_\rho u_\nu L_{\mu\rho}+(\tfrac{1}{4}\eta-\mathrm{i})(4u_\mu u_\nu-\delta_{\mu\nu}). \tag{4.8}$$

By using the transformation (3.4), where $u=1$, the sl(4, $\mathbb{R}$) generators can be expressed in terms of the Euler angles α, β, γ. The results for J_0, $J_\pm$, K_0, $K_\pm$ are given in (3.13), while those for $T_{\sigma\tau}$ are

$$\begin{aligned}
T_{\pm1,\pm1}&=\tfrac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(\alpha-\gamma)}[\pm\partial_\alpha-\mathrm{i}\sin\beta\,\partial_\beta\mp\partial_\gamma+(\mathrm{i}-\tfrac{1}{4}\eta)(1-\cos\beta)]\\
T_{\pm1,\mp1}&=\tfrac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(\alpha+\gamma)}[\pm\partial_\alpha+\mathrm{i}\sin\beta\,\partial_\beta\pm\partial_\gamma+(\mathrm{i}-\tfrac{1}{4}\eta)(1+\cos\beta)]\\
T_{\pm1,0}&=2^{-1/2}\,\mathrm{e}^{\pm\mathrm{i}\alpha}[\operatorname{cosec}\beta\,\partial_\alpha\mp\mathrm{i}\cos\beta\,\partial_\beta-\cot\beta\,\partial_\gamma\pm(\mathrm{i}-\tfrac{1}{4}\eta)\sin\beta]\\
T_{0,\pm1}&=2^{-1/2}\,\mathrm{e}^{\mp\mathrm{i}\gamma}[-\cot\beta\,\partial_\alpha\pm\mathrm{i}\cos\beta\,\partial_\beta+\operatorname{cosec}\beta\,\partial_\gamma\mp(\mathrm{i}-\tfrac{1}{4}\eta)\sin\beta]\\
T_{0,0}&=-\mathrm{i}\sin\beta\,\partial_\beta-(\mathrm{i}-\tfrac{1}{4}\eta)\cos\beta.
\end{aligned} \tag{4.9}$$

It can be easily checked that these operators satisfy the commutation relations (4.5) and (4.6), as well as the hermiticity property

$$(T_{\sigma\tau})^\dagger=(-1)^{\sigma+\tau}T_{-\sigma,-\tau} \tag{4.10}$$

with respect to the measure $\sin\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma$.

It is now straightforward to determine the action of the operators (4.9) on the hyperspherical harmonics, written in terms of Euler angles. Application of the Wigner-Eckart theorem with respect to su(2)⊕su(2) indeed leads to the relation

$$\begin{aligned}
T_{\sigma\tau}Y_{2j,m',m}(\alpha,\beta,\gamma)
&=\sum_{j'=j-1}^{j+1}\langle j'\|T\|j\rangle\langle j\,m', 1\,\sigma|j'\,m'+\sigma\rangle\langle j\,m, 1\,\tau|j'\,m+\tau\rangle\\
&\quad\times Y_{2j',m'+\sigma,m+\tau}(\alpha,\beta,\gamma)
\end{aligned} \tag{4.11}$$

where $\langle j'\|T\|j\rangle$ denotes a reduced matrix element. From (3.5) and the known expression of the rotation matrices in terms of α, β, γ (Biedenharn and Louck 1981), the latter is easily calculated for $j'=j+1$, j and $j-1$. By setting $m'=m=j$, and $\sigma=\tau=1$ or 0 in (4.11), we indeed obtain

$$\langle j+1\|T\|j\rangle=[\mathrm{i}(j+1)-\tfrac{1}{4}\eta][(2j+1)/(2j+3)]^{1/2} \tag{4.12a}$$

and

$$\langle j\|T\|j\rangle=-\tfrac{1}{4}\eta. \tag{4.12b}$$

From (4.10) and (4.12*a*), we also get

$$\langle j-1\|T\|j\rangle=-(\mathrm{i}j+\tfrac{1}{4}\eta)[(2j+1)/(2j-1)]^{1/2}. \tag{4.12c}$$

Equations (3.5) and (4.11) finally lead to the following result for the action of $T_{\sigma\tau}$ on the rotation matrices:

$$T_{\sigma\tau}D^{j*}_{m'm}(\alpha,\beta,\gamma)=(-1)^{1-\tau}(2j+1)^{-1/2}\sum_{j'=j-1}^{j+1}(2j'+1)^{1/2}\langle j'\|T\|j\rangle\langle j\,m',1\,\sigma|j'\,m+\sigma\rangle$$
$$\times\langle j\,m,1-\tau|j'\,m-\tau\rangle D^{j'*}_{m'+\sigma,m-\tau}(\alpha,\beta,\gamma). \tag{4.13}$$

By taking (4.12) into account and by replacing the SU(2) Wigner coefficients by their value, one can obtain explicit expressions for the action of the operators $T_{\sigma\tau}$ on the rotation matrices. In appendix 1, it is shown that such expressions lead to differential equations and recursion relations for the functions $d^{j}_{m'm}(\beta)$, some of which were derived before by other procedures.

From (3.15) and (4.13) it is obvious that, with respect to sl(4, $\mathbb{R}$), the set of rotation matrices separates into two subsets, corresponding to all integral or half-integral values of j, respectively. Both carry an sl(4, $\mathbb{R}$) unitary irrep of the ladder series $\mathfrak{D}^{\text{ladd}}(j_0, j_0;\ \eta)$, characterised by a real parameter η, and the minimum j value, $\min(j)=j_0$. Their (j,j) or $[N0]$ content is

$$\mathfrak{D}^{\text{ladd}}(0,0;\ \eta):\ \{(j,j)\}=\{(0,0),(1,1),(2,2),\ldots\}$$
$$\{[N0]\}=\{[00],[20],[40],\ldots\} \tag{4.14a}$$

and

$$\mathfrak{D}^{\text{ladd}}(\tfrac{1}{2},\tfrac{1}{2};\ \eta):\{(j,j)\}=\{(\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{3}{2},\tfrac{3}{2}),(\tfrac{5}{2},\tfrac{5}{2}),\ldots\}$$
$$\{[N0]\}=\{[10],[30],[50],\ldots\} \tag{4.14b}$$

respectively. Note that, for $\mathfrak{D}^{\text{ladd}}(0,0;\ \eta)$, there is a discrepancy between the results given by Ne'eman and Šijački (1979) and those of Šijački and Ne'eman (1985). Equation (4.14a) agrees with the former.

In analogy with su(4), for sl(4, $\mathbb{R}$) we may define three independent Casimir operators G_2, G_3 and G_4, of degree 2, 3 and 4 with respect to the generators, respectively. As shown in appendix 2, they are given by

$$G_2=\tfrac{1}{4}(L_{\mu\nu}L_{\mu\nu}-S_{\mu\nu}S_{\mu\nu}) \tag{4.15a}$$

$$G_3=\tfrac{1}{4}\mathrm{i}(3S_{\mu\nu}L_{\nu\xi}L_{\xi\mu}+S_{\mu\nu}S_{\nu\xi}S_{\xi\mu}) \tag{4.15b}$$

$$G_4=\tfrac{1}{8}[3(L_{\mu\nu}L_{\nu\xi}L_{\xi\rho}L_{\rho\mu}+4S_{\mu\nu}S_{\nu\xi}L_{\xi\rho}L_{\rho\mu}+2S_{\mu\nu}L_{\nu\xi}S_{\xi\rho}L_{\rho\mu}+S_{\mu\nu}S_{\nu\xi}S_{\xi\rho}S_{\rho\mu})$$
$$-24G_2^2-32G_2-120L_{\mu\nu}L_{\mu\nu}]. \tag{4.15c}$$

When replacing $L_{\mu\nu}$ and $S_{\mu\nu}$ by their realisations (3.8) and (4.8), and taking into account that $u=1$, after some straightforward but lengthy calculations, it can be shown that, on the unit sphere S^3, all three Casimir operators assume unique numerical values, given by

$$G_2=-3[1+(\tfrac{1}{4}\eta)^2] \tag{4.16a}$$

$$G_3=-\tfrac{1}{2}\mathrm{i}\eta G_2=\tfrac{3}{2}\mathrm{i}\eta[1+(\tfrac{1}{4}\eta)^2] \tag{4.16b}$$

$$G_4=\tfrac{1}{2}G_2(G_2+4)=-\tfrac{3}{2}[1+(\tfrac{1}{4}\eta)^2][1-3(\tfrac{1}{4}\eta)^2]. \tag{4.16c}$$

By proceeding as in the sl(3, $\mathbb{R}$) case (Weaver and Biedenharn 1972, Biedenharn *et al* 1972, Šijački 1975), these values may be compared with the eigenvalues of the corresponding su(4) Casimir operators, given in appendix 2. This enables us to assign the (generalised) Young pattern labels $[pqr0]$, $p=-2-\tfrac{1}{2}\mathrm{i}\eta$ and $q=r=0$, to both irreps (4.14).

By using (2.11), it is straightforward to go from the sl(4, ℝ) algebra corresponding to the rotation matrices to that associated with the Pöschl–Teller potentials. In addition to the operators $\tilde{L}_{\mu\nu}$, or $\tilde{J}_0$, $\tilde{J}_\pm$, $\tilde{K}_0$, $\tilde{K}_\pm$, defined in (3.17) and (3.18) respectively, the latter includes the generators

$$\tilde{S}_{\mu\nu} = (\sin\beta)^{1/2} S_{\mu\nu} (\sin\beta)^{-1/2} \tag{4.17}$$

where use must be made of (3.4) and (4.8). The operators $\tilde{T}_{\sigma\tau}$, as obtained from $T_{\sigma\tau}$ by a similar procedure, can be written as

$$\begin{aligned}
\tilde{T}_{\pm1,\pm1} &= \tfrac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(\alpha-\gamma)}[\mathrm{i}(2a)^{-1}\sin(2ax)\partial_x \pm \partial_\alpha \mp \partial_\gamma + \mathrm{i} - \tfrac{1}{4}\eta + \tfrac{1}{4}(2\mathrm{i}-\eta)\cos(2ax)] \\
\tilde{T}_{\pm1,\mp1} &= \tfrac{1}{2}\,\mathrm{e}^{\pm\mathrm{i}(\alpha+\gamma)}[-\mathrm{i}(2a)^{-1}\sin(2ax)\partial_x \pm \partial_\alpha \pm \partial_\gamma + \mathrm{i} - \tfrac{1}{4}\eta - \tfrac{1}{4}(2\mathrm{i}-\eta)\cos(2ax)] \\
\tilde{T}_{\pm1,0} &= 2^{-1/2}\,\mathrm{e}^{\pm\mathrm{i}\alpha}[\mp\mathrm{i}(2a)^{-1}\cos(2ax)\partial_x + \operatorname{cosec}(2ax)\partial_\alpha + \cot(2ax)\partial_\gamma \\
&\qquad \pm\tfrac{1}{2}\mathrm{i}\operatorname{cosec}(2ax) \pm \tfrac{1}{4}(2\mathrm{i}-\eta)\sin(2ax)] \\
\tilde{T}_{0,\pm1} &= 2^{-1/2}\,\mathrm{e}^{\mp\mathrm{i}\gamma}[\pm\mathrm{i}(2a)^{-1}\cos(2ax)\partial_x + \cot(2ax)\partial_\alpha + \operatorname{cosec}(2ax)\partial_\gamma \\
&\qquad \mp\tfrac{1}{2}\mathrm{i}\operatorname{cosec}(2ax) \mp \tfrac{1}{4}(2\mathrm{i}-\eta)\sin(2ax)] \\
\tilde{T}_{0,0} &= \mathrm{i}(2a)^{-1}\sin(2ax)\partial_x + \tfrac{1}{4}(2\mathrm{i}-\eta)\cos(2ax).
\end{aligned} \tag{4.18}$$

It can be checked that the operators $\tilde{J}_0$, $\tilde{J}_\pm$, $\tilde{K}_0$, $\tilde{K}_\pm$ and $\tilde{T}_{\sigma\tau}$ satisfy relations similar to (4.5) and (4.6), and that $\tilde{T}_{\sigma\tau}$ fulfils the hermiticity property (4.10) with respect to the measure $\mathrm{d}x\,\mathrm{d}\alpha\,\mathrm{d}\gamma$.

The action of the operators $\tilde{T}_{\sigma\tau}$ on the extended wavefunctions (2.11) can be derived from that of $T_{\sigma\tau}$ on the (complex conjugate) Wigner rotation matrices $D^{j*}_{m'm}(\alpha,\beta,\gamma)$, given in (4.13). The results are

$$\begin{aligned}
&\tilde{T}_{\sigma\tau}\Psi_n^{(m',m)}(x,\alpha,\gamma) \\
&\quad = (-1)^{1-\tau}\sum_{n'=n-\sigma-1}^{n-\sigma+1} c_{n'}(m'+n)\langle m'+n\ \ m', 1\ \ \sigma | m'+n'+\sigma\ \ m'+\sigma\rangle \\
&\qquad \times\langle m'+n\ \ m, 1\ \ -\tau | m'+n'+\sigma\ \ m-\tau\rangle \Psi_{n'}^{(m'+\sigma,m-\tau)}(x,\alpha,\gamma)
\end{aligned} \tag{4.19}$$

where $c_{n'}(m'+n)$ is given by

$$\begin{aligned}
&c_{n'}(m'+n) \\
&\quad = -[\mathrm{i}(m'+n)+\tfrac{1}{4}\eta][(2m'+2n+1)/(2m'+2n-1)]^{1/2} && \text{if } n'=n-\sigma-1 \\
&\quad = -\tfrac{1}{4}\eta && \text{if } n'=n-\sigma \\
&\quad = [\mathrm{i}(m'+n+1)-\tfrac{1}{4}\eta][(2m'+2n+1)/(2m'+2n+3)]^{1/2} && \text{if } n'=n-\sigma+1.
\end{aligned} \tag{4.20}$$

From (4.19), it follows that the operator $\tilde{T}_{00}$ generates transitions between the extended wavefunctions $\Psi_n^{(m',m)}$ and $\Psi_{n'}^{(m',m)}$, $n'=n-1, n, n+1$, corresponding to the same potential. Energy raising and lowering operators $\mathscr{R}_n$, $\mathscr{L}_n$, corresponding to the ladder operators of the factorisation method (Infeld and Hull 1951), can be easily constructed in the enveloping algebra of sl(4, ℝ). A possible choice is

$$\mathscr{R}_n = (\mathscr{L}_{n+1})^\dagger = (n+1)(2m'+n+1)\tilde{T}_{0,0} + (2m'+n+1)\tilde{T}_{+1,0}\tilde{J}_{-1} - (n+1)\tilde{T}_{-1,0}\tilde{J}_{+1} \tag{4.21}$$

where

$$\begin{aligned}
&\mathscr{R}_n\Psi_n^{(m',m)}(x,\alpha,\gamma) \\
&\quad = [-\mathrm{i}(m'+n+1)+\tfrac{1}{4}\eta][(n+1)(2m'+n+1)(2m'+2n+1)(m'-m+n+1) \\
&\qquad \times(m'+m+n+1)]^{1/2}(2m'+2n+3)^{-1/2}\Psi_{n+1}^{(m',m)}(x,\alpha,\gamma)
\end{aligned} \tag{4.22}$$

and

$$\mathscr{L}_n \Psi_n^{(m',m)}(x, \alpha, \gamma)$$
$$= [\mathrm{i}(m'+n)+\tfrac{1}{4}\eta][n(2m'+n)(2m'+2n-1)(m'-m+n)(m'+m+n)]^{1/2}$$
$$\times (2m'+2n+1)^{-1/2}\Psi_{n-1}^{(m',m)}(x, \alpha, \gamma). \tag{4.23}$$

Raising and lowering operators $\mathscr{R}$, $\mathscr{L}$, which do not refer to the index n of the function being operated upon, can be obtained from (4.21) by considering the operator

$$\tilde{J} \equiv (\tilde{\boldsymbol{J}}^2+\tfrac{1}{4})^{1/2}-\tfrac{1}{2} \tag{4.24}$$

whose eigenvalue, corresponding to $\Psi_n^{(m',m)}$, is $j = m'+n$. They are given by

$$\mathscr{R} = \mathscr{L}^\dagger = \tilde{T}_{0,0}(\tilde{J}-\tilde{J}_0+1)(\tilde{J}+\tilde{J}_0+1)+\tilde{T}_{+1,0}\tilde{J}_{-1}(\tilde{J}+\tilde{J}_0+1)-\tilde{T}_{-1,0}\tilde{J}_{+1}(\tilde{J}-\tilde{J}_0+1) \tag{4.25}$$

and act on $\Psi_n^{(m',m)}$ in the same way as $\mathscr{R}_n$ and $\mathscr{L}_n$, respectively.

5. Conclusion

We have proved that all the eigenstates, corresponding to the family of Pöschl–Teller potentials with quantised potential strengths (m', m) differing by integers, belong to the carrier space of a single sl(4, $\mathbb{R}$) ladder unitary irrep. For integral values of m' and m, this irrep is $\mathfrak{D}^{\text{ladd}}(0, 0; \eta)$, while for half-integral values, it is $\mathfrak{D}^{\text{ladd}}(\tfrac{1}{2}, \tfrac{1}{2}; \eta)$. In terms of the original strengths κ and λ, this means that all the eigenstates, corresponding to potentials with half-integral values of κ and λ such that $\kappa+\lambda$ is odd (even) and $\kappa-\lambda$ even (odd), belong to the carrier space of $\mathfrak{D}^{\text{ladd}}(0, 0; \eta)$ ($\mathfrak{D}^{\text{ladd}}(\tfrac{1}{2}, \tfrac{1}{2}; \eta)$). As stressed in § 2, the carrier spaces of both ladder irreps actually contain four copies of the potential family eigenstates in addition to some unphysical states.

In the following paper (Quesne 1988), we shall specialise the present analysis to the subfamily of one-parameter symmetrical Pöschl–Teller potentials and contrast it with another approach. In a forthcoming publication we also plan to construct dynamical potential algebras for the second family of Pöschl–Teller potentials, as well as for the Rosen–Morse and Eckart potentials.

Appendix 1. Differential equations and recursion relations for $d^j_{m'm}(\beta)$

The purpose of this appendix is to illustrate the usefulness of the set of relations (4.13) by deriving both differential equations and recursion relations for $d^j_{m'm}(\beta)$ from their explicit form.

Since η may take any real value, the η-independent and η-dependent terms may be separately equated on both sides of (4.13). After making the substitution

$$D^{j*}_{m'm}(\alpha, \beta, \gamma) = \exp(\mathrm{i}m'\alpha)d^j_{m'm}(\beta)\exp(\mathrm{i}m\gamma) \tag{A1.1}$$

and taking (4.9) and (4.12) into account, the η-independent terms give rise to differential equations for $d^j_{m'm}(\beta)$, while the η-dependent ones lead to recursion relations for the same functions.

For $\sigma = \tau = \pm 1$, for instance, we obtain the equations

$$(2j+1)[-\sin\beta\, d_\beta \pm (m'-m)+1-\cos\beta]d^j_{m'm}(\beta)$$
$$= [(j\pm m'+1)(j\pm m'+2)(j\mp m+1)(j\mp m+2)]^{1/2}d^{j+1}_{m'\pm1,m\mp1}(\beta)$$
$$-[(j\mp m'-1)(j\mp m')(j\pm m-1)(j\pm m)]^{1/2}d^{j-1}_{m'\pm1,m\mp1}(\beta) \tag{A1.2}$$

and the relations

$$
\begin{aligned}
j(j+1)(2j+1)(1-\cos\beta)d^j_{m'm}(\beta)
&= j[(j\pm m'+1)(j\pm m'+2)(j\mp m+1)(j\mp m+2)]^{1/2}d^{j+1}_{m'\pm1,m\mp1}(\beta)\\
&\quad -(2j+1)[(j\mp m')(j\pm m'+1)(j\pm m)(j\mp m+1)]^{1/2}d^{j}_{m'\pm1,m\mp1}(\beta)\\
&\quad +(j+1)[(j\mp m'-1)(j\mp m')(j\pm m-1)(j\pm m)]^{1/2}d^{j-1}_{m'\pm1,m\mp1}(\beta). \qquad \text{(A1.3)}
\end{aligned}
$$

Seven additional differential equations and recursion relations can be similarly derived for the remaining values of (σ, τ): $(\sigma=\pm1, \tau=\mp1)$, $(\sigma=\pm1, \tau=0)$, $(\sigma=0, \tau=\pm1)$ and $(\sigma=0, \tau=0)$.

The nine differential equations so obtained can of course be proved by other procedures. For instance, that corresponding to $\sigma=\tau=0$,

$$
\begin{aligned}
(2j+1)(\sin\beta\, d_\beta+\cos\beta)d^j_{m'm}(\beta)
&= [(j-m'+1)(j+m'+1)(j-m+1)(j+m+1)]^{1/2}d^{j+1}_{m'm}(\beta)\\
&\quad -[(j-m')(j+m')(j-m)(j+m)]^{1/2}d^{j-1}_{m'm}(\beta) \qquad \text{(A1.4)}
\end{aligned}
$$

results from combining two differential equations given by Schneider and Wilson (1979).

On the other hand, all nine recursion relations are but the explicit form of the Clebsch–Gordan series

$$
d^j_{m'm}(\beta)d^\lambda_{\mu'\mu}(\beta)=\sum_{j'}\langle j\ m',\lambda\ \mu'|j'\ m'+\mu\rangle\langle j\ m,\lambda\ \mu|j'\ m+\mu\rangle d^{j'}_{m'+\mu',m+\mu}(\beta) \qquad \text{(A1.5)}
$$

where $\lambda=1$, $\mu'=\sigma$ and $\mu=-\tau$.

Appendix 2. Casimir operators of sl(4, $\mathbb{R}$)

The purpose of this appendix is to prove equations (4.15*a*), (4.15*b*) and (4.15*c*), giving explicit expressions for the sl(4, $\mathbb{R}$) Casimir operators G_k, $k=2, 3, 4$.

The su(4) algebra is spanned by the operators

$$
E_{\mu\nu}=\mathscr{E}_{\mu\nu}-\tfrac{1}{4}\delta_{\mu\nu}\mathscr{E}_{\rho\rho} \qquad \mu,\nu=1,\ldots,4 \qquad \text{(A2.1)}
$$

where $\mathscr{E}_{\mu\nu}=(\mathscr{E}_{\nu\mu})^\dagger$ are u(n) generators, satisfying the commutation relations

$$
[\mathscr{E}_{\mu\nu},\mathscr{E}_{\mu'\nu'}]=\delta_{\nu\mu'}\mathscr{E}_{\mu\nu'}-\delta_{\mu\nu'}\mathscr{E}_{\mu'\nu}. \qquad \text{(A2.2)}
$$

By separating the second-rank tensor $E_{\mu\nu}$ into its antisymmetrical and symmetrical parts, we obtain the operators

$$
L_{\mu\nu}=-L_{\nu\mu}=(L_{\mu\nu})^\dagger=-\mathrm{i}(E_{\mu\nu}-E_{\nu\mu}) \qquad \text{(A2.3)}
$$

and

$$
S_{\mu\nu}=S_{\nu\mu}=(S_{\mu\nu})^\dagger=E_{\mu\nu}+E_{\nu\mu} \qquad \text{(A2.4)}
$$

where $S_{\mu\mu}=0$. They form an alternative basis of su(4), and their commutation relations are given by

$$
\begin{aligned}
[L_{\mu\nu},L_{\mu'\nu'}]&=\mathrm{i}(\delta_{\mu\mu'}L_{\nu\nu'}-\delta_{\mu\nu'}L_{\nu\mu'}-\delta_{\nu\mu'}L_{\mu\nu'}+\delta_{\nu\nu'}L_{\mu\mu'})\\
[L_{\mu\nu},S_{\mu'\nu'}]&=\mathrm{i}(\delta_{\mu\mu'}S_{\nu\nu'}+\delta_{\mu\nu'}S_{\nu\mu'}-\delta_{\nu\mu'}S_{\mu\nu'}-\delta_{\nu\nu'}S_{\mu\mu'}) \qquad \text{(A2.5)}\\
[S_{\mu\nu},S_{\mu'\nu'}]&=\mathrm{i}(\delta_{\mu\mu'}L_{\nu\nu'}+\delta_{\mu\nu'}L_{\nu\mu'}+\delta_{\nu\mu'}L_{\mu\nu'}+\delta_{\nu\nu'}L_{\mu\mu'}).
\end{aligned}
$$

Comparison between (A2.5) and the commutation relations (3.9) and (4.1) of the sl(4, ℝ) generators shows that the latter can be obtained from the former by changing $S_{\mu\nu}$ into $-\mathrm{i}S_{\mu\nu}$. Hence, the same substitution, carried out in the su(4) Casimir operators, will yield those of sl(4, ℝ).

A set of independent su(4) Casimir operators is given by

$$G_k = \sum_{\pi} (-1)^{k-1+\pi} E_{\mu_1,\pi(\mu_1)} E_{\mu_2,\pi(\mu_2)} \ldots E_{\mu_k,\pi(\mu_k)} \qquad k = 2, 3, 4 \qquad \text{(A2.6)}$$

where the summation is carried out over the $k!$ permutations π of the indices $\mu_1, \mu_2, \ldots, \mu_k$. They can be expressed as

$$G_2 = \bar{G}_2 \qquad G_3 = 2\bar{G}_3 - 4\bar{G}_2$$
$$G_4 = 6\bar{G}_4 - 24\bar{G}_3 - 3\bar{G}_2^2 + 20\bar{G}_2 \qquad \text{(A2.7)}$$

in terms of the commonly used operators

$$\bar{G}_k = E_{\mu_1\mu_2} E_{\mu_2\mu_3} \ldots E_{\mu_k\mu_1} \qquad k = 2, 3, 4. \qquad \text{(A2.8)}$$

From the well known eigenvalues of $\bar{G}_k$ (Louck 1970), it can be shown that those of G_k, corresponding to an su(4) irrep characterised by the Young pattern labels $[pqr0]$, are given by

$$g_2(p, q, r) = \tfrac{1}{4}(3p^2 - 2pq - 2pr + 3q^2 - 2qr + 3r^2 + 12p + 4q - 4r)$$
$$g_3(p, q, r) = \tfrac{3}{4}(p - q - r)(p - q + r + 2)(p + q - r + 4)$$
$$\begin{aligned} g_4(p, q, r) = \tfrac{1}{32}(&9p^4 - 12p^3q - 12p^3r - 42p^2q^2 + 60p^2qr - 42p^2r^2 - 12pq^3 \\ &+ 60pq^2r + 60pqr^2 - 12pr^3 + 9q^4 - 12q^3r - 42q^2r^2 - 12qr^3 \\ &+ 9r^4 + 72p^3 - 216p^2q - 72p^2r - 264pq^2 + 720pqr \\ &- 168pr^2 + 24q^3 + 24q^2r - 24qr^2 - 24r^3 + 192p^2 - 896pq + 256pr \\ &- 192q^2 + 1024qr - 192r^2 + 192p - 704q + 704r). \end{aligned} \qquad \text{(A2.9)}$$

Substituting $\frac{1}{2}(S_{\mu\nu} + \mathrm{i}L_{\mu\nu})$ for $E_{\mu\nu}$ in (A2.6) and using the commutation relations (A2.5), we obtain for G_k, $k = 2, 3, 4$, the alternative expressions

$$G_2 = \tfrac{1}{4}(L_{\mu\nu}L_{\mu\nu} + S_{\mu\nu}S_{\mu\nu}) \qquad G_3 = \tfrac{1}{4}(-3S_{\mu\nu}L_{\nu\xi}L_{\xi\mu} + S_{\mu\nu}S_{\nu\xi}S_{\xi\mu})$$
$$\begin{aligned} G_4 = \tfrac{1}{8}[3(&L_{\mu\nu}L_{\nu\xi}L_{\xi\rho}L_{\rho\mu} - 4S_{\mu\nu}S_{\nu\xi}L_{\xi\rho}L_{\rho\mu} - 2S_{\mu\nu}L_{\nu\xi}S_{\xi\rho}L_{\rho\mu} \\ &+ S_{\mu\nu}S_{\nu\xi}S_{\xi\rho}S_{\rho\mu}) - 24G_2^2 - 32G_2 - 120L_{\mu\nu}L_{\mu\nu}]. \end{aligned} \qquad \text{(A2.10)}$$

It only remains to replace $S_{\mu\nu}$ by $-\mathrm{i}S_{\mu\nu}$ in (A2.10) to get the searched-for relations (4.15).

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